MODULE *tictactoe*

EXTENDS *Naturals*

VARIABLES
  *board*, *board*[1 . . 3][1 . . 3] A 3$x$3 tic-tac-toe board
  *nextTurn* who goes next

$Pieces \triangleq \{\text{"X"}, \text{"O"}, \text{"\_"}\}$ "_" represents a blank square

$Init \triangleq$
  $\land\ nextTurn = \text{"X"}$ $X$ always goes first
  Every space in the board states blank
  $\land\ board = [i \in 1\,..\,3 \mapsto [j \in 1\,..\,3 \mapsto \text{"\_"}]]$

$Move(player) \triangleq$
  $\exists\, i \in 1\,..\,3 : \exists\, j \in 1\,..\,3 :$ There exists a position on the board
    $\land\ board[i][j] = \text{"\_"}$ Where the board is currently empty
    The future state of board is the same, except a piece is in that spot
    $\land\ board' = [board \text{ EXCEPT } ![i][j] = player]$

$MoveX \triangleq$
  $\land\ nextTurn = \text{"X"}$ Only enabled on $X$'s turn
  $\land\ Move(\text{"X"})$
  $\land\ nextTurn' = \text{"O"}$ The future state of next turn is $O$

$MoveO \triangleq$
  $\land\ nextTurn = \text{"O"}$ Only enabled on $O$'s turn
  $\land\ Move(\text{"O"})$
  $\land\ nextTurn' = \text{"X"}$ The future state of next turn is $X$

Every state, $X$ will move if $X$'s turn, $O$ will move on $O$'s turn
$Next \triangleq MoveX \lor MoveO$

A description of every possible game of tic-tac-toe
will play until the board fills up, even if someone won
$Spec \triangleq Init \land \Box[Next]_{\langle board,\, nextTurn \rangle}$

Invariants: The things we are checking for.

$WinningPositions \triangleq \{$
  Horizonal wins
  $\{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\}$,
  $\{\langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle\}$,
  $\{\langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle\}$,
  Vertical wins
  $\{\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle\}$,

$\{\langle 1, 2\rangle, \langle 2, 2\rangle, \langle 3, 2\rangle\},$
$\{\langle 1, 3\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\},$

Diagonal wins

$\{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\},$
$\{\langle 3, 1\rangle, \langle 2, 2\rangle, \langle 1, 3\rangle\}$
$\}$

$Won(player) \triangleq$

A player has won if there exists a winning position

$\exists\, winningPosition \in WinningPositions :$

Where all the needed spaces

$\forall\, neededSpace \in winningPosition :$

are occupied by one player

$board[neededSpace[1]][neededSpace[2]] = player$

$XHasNotWon \triangleq \neg Won(\text{"X"})$
$OHasNotWon \triangleq \neg Won(\text{"O"})$

$BoardFilled \triangleq$

There does not exist

$\neg \exists\, i \in 1 \,..\, 3, j \in 1 \,..\, 3 :$

an empty space

$\text{LET } space \triangleq board[i][j] \text{ IN}$
$space = \text{"\_"}$

It's not a stalemate if one player has won or the board is not filled

$NotStalemate \triangleq$
$\lor\, Won(\text{"X"})$
$\lor\, Won(\text{"O"})$
$\lor\, \neg BoardFilled$